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THE ROHLIN TOWER THEOREM AND HYPER-FINITENESS FOR ACTIONS OF CONTINUOUS GROUPS

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ABSTRACT

We prove the Rohlin tower theorem for free measure preserving actions of locally compact second countable solvable groups and almost connected amenable groups. This theorem was known for l.c.s.c. abelian groups and was recently extended by Ornstein and Weiss to discrete solvable groups. We extend their methods to the continuous case, using the structure theory of the class of groups under consideration. As a corollary we obtain that free actions of such groups generate hyperfinite equivalence relations.

Introduction

Let T be an aperiodic measure preserving automorphism of a probability space X. The Rohlin tower theorem states that for any positive integer n and any $\varepsilon > 0$ there is a Borel set $E \subseteq X$ such that the sets $E, TE, \cdots, T^nE$ are disjoint and fill X to within ε. This result is of crucial importance for two results: Dye's theorem [2], that is the hyperfiniteness of the equivalence relations generated by aperiodic transformations; and the isomorphism theorem for Bernouilli shifts [13]. Generalizations of Rohlin's theorem to non-measure preserving transformations and more general groups have been given by various authors; in particular Krieger [7] extended it to measure preserving actions of discrete abelian groups and Lind [8], to n-dimensional flows. Dye [3] proved the hyperfiniteness of equivalence relations generated by free actions of countable abelian groups, and Connes and Krieger [1] proved hyperfiniteness for free measure class preserving actions of countable solvable groups. Ornstein and Weiss [14] have given a more constructive proof of the Rohlin theorem for solvable groups. (For a precise statement of the Rohlin theorem see §1 below.)

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The main result of this paper is that the Rohlin theorem holds for finite extensions of continuous solvable groups and for almost connected amenable groups. Our techniques are an extension of those used by Ornstein and Weiss in the discrete case.

In §1 we give a precise statement of the Rohlin theorem and prove some general results about Rohlin towers. We show that if $K \to G \to G/K$ is an exact sequence of locally compact second countable groups where K is compact and G/K is an R-group (i.e. the Rohlin theorem holds in G/K) then so is G. §2 contains a detailed proof that if H is an R-group then so are split extensions of H by $\mathbf{R}$. In §3 we show how to modify this to deal with arbitrary extensions of H by $\mathbf{R}$ or a compact group. As corollaries we obtain the results stated above. We are indebted to J. Feldman for pointing out how to obtain Corollary 1.11.

In §4 we show how to extend the notion of hyperfiniteness to uncountable equivalence relations (cf. also [4]) and show that free actions of R-groups generate hyperfinite equivalence relations. Finally we show that any free action of a continuous R-group is weakly equivalent to a flow.

§1. Rohlin towers

Let G be a locally compact second countable (l.c.s.c.) group and let X be a standard Borel G space (cf. [9]) with an invariant probability measure μ. Let $F \subseteq G$ be a Borel set. An F-base in X is a Borel set $V \subseteq X$ such that FV is Borel, $\mu(FV) > 0$, and such that the sets $fV, f \in F$, are disjoint. An F tower $\bar{V} \subseteq X$ is a set $\bar{V} = FV$, where V is an F base. The following result shows that the measure structure of a tower is exactly as expected.

PROPOSITION 1.1. *Let G be a l.c.s.c. group and let X be a standard Borel G space with invariant probability measure μ. Let λ be a left Haar measure on G, and let $F \subseteq G$ be a Borel set with $\lambda(F) > 0$. Let $\bar{V}$ be an F tower in X on a base V, and let $F \times V$ have the Borel structure induced from $G \times X$. Then $P: F \times V \to \bar{V}$, $P(f, v) = fv$, is a Borel isomorphism and there is a measure ν on V so that $P_*(\lambda \times \nu) = \mu |_{\bar{V}}$. ν will be called the measure induced on V by $\bar{V}$.*

PROOF. P is a Borel bijection and $\bar{V}$ is standard, hence P is an isomorphism (cf. [9]).

Let $p: \bar{V} \to V$ be projection, $p(fv) = v$, and set $\nu = p_* \bar{\mu} / \lambda(F)$ where $\bar{\mu} = \mu |_{\bar{V}}$. Let $\bar{\mu} = \int_V \mu_v d\nu(v)$ be a decomposition of $\bar{\mu}$ with respect to p. Choose a countable dense set $\{g_i : i \in \mathbf{Z}\}$ in G such that, up to a null set, $G = \bigcup_{i=1}^{\infty} g_i F$. For $g \in G$, let $\tilde{g}: F \to gF$ be the map $\tilde{g}(f) = gf$. Then for ν a.a. $v \in V$, $\tilde{g}_* \mu_v$ is a measure on gF. We will show that for ν a.a. $v \in V$,

$$\tilde{g}_{i*}\mu_v = \tilde{g}_{j*}\mu_v \quad \text{on} \quad g_iF \cap g_jF \quad \forall i, j \in \mathbf{Z} \tag{1}$$

or equivalently

$$\widetilde{g_j^{-1}g_i}_*\mu_v = \mu_v \quad \text{on} \quad g_j^{-1}g_iF \cap F \quad \forall i, j \in \mathbf{Z}. \tag{2}$$

Suppose $h \in G$ and $hF \cap F = E \neq \emptyset$. Then $\tilde{h}: h^{-1}E \to E$ and $h^{-1}E, E \subseteq F$. We also have $\bar{h}: h^{-1}EV \to EV$, $\bar{h}(x) = hx$, and $\bar{h}_*\mu\,|_{h^{-1}EV} = \mu\,|_{EV}$. Thus

$$\int_V \mu_v\,|_{EV} d\nu(v) = \mu\,|_{EV} = \bar{h}_*\mu\,|_{h^{-1}EV}$$

$$= \bar{h}_* \int_V \mu_v\,|_{h^{-1}EV} d\nu(v) = \int_V \bar{h}_*\mu_v\,|_{EV} d\nu(v).$$

Therefore $\mu_v\,|_{EV} = \tilde{h}_*\mu_v\,|_{EV}$ for ν a.a. $v \in V$, and so (2) holds outside a ν null set N in V.

If $v \notin N$ then by (1) we may define a measure $\bar{\mu}_v$ on G by $\bar{\mu}_v\,|_{g_iF} = \tilde{g}_{i*}\mu_v$. $\bar{\mu}_v$ is clearly invariant under left translation by each g_i. The weak continuity of the left regular representation ensures $\bar{\mu}_v$ is left invariant, so that $\bar{\mu}_v = f(v)\lambda$ where $f: V \to \mathbf{R}^+$ is measurable. $f(v) = \mu_v(F)/\lambda(F)$, and for $W \in \mathcal{B}(V)$,

$$\lambda(F)\int_W d\nu(v) = \bar{\mu}(FW) = \int_W \mu_v(F)d\nu(v) = \int \lambda(F)f(v)d\nu(v)$$

so that $f(v) = 1$ a.a. $v \in V$, and $\bar{\mu} = \lambda\,|_F \times \nu$.

PROPOSITION 1.2. *In the situation of Proposition* 1.1, *let $Q \subseteq G$ be an open set and let $\bar{V}$, $\bar{W}$ be Q towers with induced measures η, ν respectively on V, W. Let $T: V \to W$ be a measurable bijection of the form $T(v) = \beta(v)$, $\beta(v) \in G$. Then $\Delta(\beta(v))dT_*\eta(Tv) = d\nu(Tv)$ where Δ is the modular function of G.*

PROOF. Let $\{K_n\}$ be an increasing sequence of compact sets with $\bigcup K_n = G$. Let $V_n = \{v \in V: Q\beta(v) \subseteq K_n\}$. Then $\bigcup V_n = V$. It is sufficient to prove the result for V_n, $T(V_n)$, i.e. we may assume $Q\beta(v) \subseteq K$ for some compact set K. Just as in the proof of Forrest's theorem [5] II we may find $\varepsilon > 0$ and an open ball $B_\varepsilon \subseteq X$ such that $\eta(B_\varepsilon) > 0$ and so that $x \sim y \Leftrightarrow x = gy$, $g \in K$, is an equivalence relation on B_ε, in which each orbit is a compact set in X. Find a measurable section F for this relation which is contained in V. Then as in [5] F is a K base in X.

If $\eta(F) \neq \eta(V)$ we may repeat the argument on $V - F$. An exhaustion argument shows that $V = \bigcup F_n$ (up to an η null set) where each F_n is a K base. It is therefore enough to see that $T_*(\eta\,|_F) = \nu\,|_{T(F)}$ when F is a K base.

Let $E = T(F)$. Say $A \subseteq KF \cap QE$. Set $A_v = \{k \in K : kv \in A\}$, $A_w = \{q \in Q : qw \in A\}$. Then $A_v = A_{T(v)}\beta(v)$. For if $a \in A$, then $a = kf = qe = q\beta(f')f'$ and $q\beta(f') \in K$, hence $f = f'$. Now

$$\mu(A) = \int_F \lambda(A_v)d\eta(v) = \int_F \lambda(A_{T(v)}\beta(v))d\eta(v) = \int_F \Delta(\beta(v))\lambda(A_{T(v)})d\eta(v)$$

$$= \int_F \Delta(\beta(v))\lambda(A_w)dT_*\eta(w)$$

and

$$\mu(A) = \int_E \lambda(A_w)d\nu(w).$$

Say $B \subseteq E$. Then $QB \subseteq KF \cap QE$. Hence, applying the above with $A = QB$, we have $\int_F \Delta(\beta(v))\lambda(Q)dT_*\eta(w) = \int_E \lambda(Q)d\nu(w)$ so that $\Delta(\beta(v))dT_*\eta(w) = d\nu(w)$, where $w = \beta(v)v$.

If G is a l.c.s.c. group acting freely on X and if $D \subseteq G$ is a compact neighborhood of the identity, the existence of D towers in X is guaranteed by Forrest's theorem [5]. We give a stronger version of this result which will be of use later.

PROPOSITION 1.3. *Let G be a l.c.s.c. group acting freely on a standard Borel space X with invariant probability measure μ. Let D_1, D_2 be relatively compact open neighborhoods of $e \in G$, $D_1 \subseteq D_2$, and let $A \in \mathcal{B}(X)$, $\mu(A) > 0$. Then there is a D_2 base $B \subset A$ such that $\mu(D_1B \cap A) > 0$.*

PROOF. We follow the method of [5]. By [10] lemma 2, we may assume that X is a separable metric space with metric d and that G acts continuously. Choose E_2 a relatively compact symmetric open neighborhood of $e \in G$ such that $D_2 \subseteq \bar{E}_2$.

For $\varepsilon > 0$, set

$$A_\varepsilon = \{x \in X : d(x, gx) \leqq \varepsilon \Rightarrow g \in E_2 \quad \text{or} \quad g \notin \bar{E}_2\bar{E}_2\}.$$

As in [5], A_ε is open and $X = \bigcup_{\varepsilon > 0} A_\varepsilon$. Pick $\alpha > 0$ such that $\mu(A_\alpha \cap A) > 0$. A_α is the union of a countable number of open balls of radius less than $\alpha/2$. Pick such a ball $S_\alpha \subseteq A_\alpha$ with $\mu(S_\alpha \cap A) > 0$. For $\delta > 0$, set $B_\delta = \{x \in X : d(x, gx) \leqq \delta \Rightarrow g \in D_1$ or $g \notin \bar{E}_2\}$. Again B_δ is open and $X = \bigcup_{\delta > 0} B_\delta$. Choose $\beta > 0$ such that $\mu(S_\alpha \cap A \cap B_\beta) > 0$. Choose a $\beta/2$ ball $S_\beta \subseteq S_\alpha \cap B_\beta$ such that $\mu(S_\beta \cap A) > 0$. Define an equivalence relation on S_β by $x \sim y \Leftrightarrow x = dy$, $d \in \bar{E}_2$. This is an equivalence relation since $\bar{E}_2$ is symmetric and since $\operatorname{diam}(S_\beta) < \alpha$ and $S_\beta \subset A_\alpha$. Since $\bar{E}_2$ is compact the quotient space $S_\beta/\sim$ is a standard Borel space (cf. [11] lemma 2). Choose a measurable section $T: S_\beta/\sim \to S_\beta$ so that if $x \in S_\beta$, and orb$x \cap A \neq \emptyset$, then $T(\bar{x}) \in A$ where $\bar{x}$ is the image of x in $S_\beta/\sim$. Set $B = T(S_\beta/\sim) \cap A$. $S_\beta \cap A \subseteq \bar{E}_2 B$, so $\mu(\bar{E}_2 B) > 0$. Say $x \in S_\beta \cap A$. Then $x = db$, $d \in \bar{E}_2$, $b \in B \cap A$, by the way in which T was chosen. $d(x, b) < \beta$ and so $d \in D_1$, $S_\beta \cap A \subseteq D_1 B$. Thus $\mu(D_1 B \cap A) > 0$. B is the required base.

In the case of a single transformation, or action of $\mathbf{Z}$, the Rohlin theorem gives the existence of $\{0, 1, \cdots, n\}$ towers in X which fill X to within any prescribed amount, for arbitrarily large n. We now formulate a Rohlin theorem for an arbitrary l.c.s.c. group, following [14]. G is a l.c.s.c. group, X a standard Borel G space with invariant probability measure μ, λ a left Haar measure on G.

Definition 1.4. Let $F \subseteq G$ be compact, and $\varepsilon > 0$. $K \subseteq G$ is F, ε (left) invariant if $\lambda(K) < \infty$ and

$$\lambda(\{k \in K : fk \in K \ \forall f \in F\}) > (1 - \varepsilon)\lambda(K).$$

A set $A \in \mathcal{B}(X)$ is F, ε invariant if

$$\mu(\{x \in A : fx \in A \ \forall f \in F\}) > (1 - \varepsilon)\mu(A).$$

Definition 1.5. A relatively compact open set $F \subseteq G$ is an R-set if for any free Borel measure preserving action of G on a standard probability space X, and for any $\varepsilon > 0$ there is an F-tower $\bar{V} \subseteq X$ with $\mu(\bar{V}) > 1 - \varepsilon$.

Definition 1.6. G is an R-group if for any compact sets $E, F \subseteq G$ and any $\varepsilon > 0$ there exists an R-set $Q \subseteq G$ such that $E \subseteq Q$ and Q is F, ε invariant. Usually we will abbreviate this by saying there exist arbitrarily large arbitrarily left invariant R-sets in G.

It is known that discrete abelian groups are R-groups [7] and that $\mathbf{R}^n \times \mathbf{Z}^m$ is an R-group [8]. In [14] it is shown that if H is a discrete R-group then so are extensions of H by discrete cyclic groups. As a corollary all solvable discrete groups are R-groups [14]. We extend this result to the continuous case: in §2 we prove

Theorem 1.7. *If H is an R-group and G a split extension of H by $\mathbf{R}$, then G is an R-group.*

In §3 we extend this to more general extensions: a piecewise continuous extension G of H by L is an extension such that for each compact set $K \subseteq L$ there is a partition $K = \bigcup_{i=1}^{n} K_i$, and a section $\alpha: L \to G$, so that $\alpha|_{K_i}$ is continuous for each i. We prove:

THEOREM 1.8. *If H is an R-group and G a piecewise continous extension of H by* $\mathbf{R}$ *or a compact group K, then G is an R-group.*

The methods of [14] and Proposition 1.1 give a similar result for extensions of H by $\mathbf{Z}$ (notice that such extensions necessarily split). We also observe the following:

PROPOSITION 1.9. *Let G be a l.c.s.c. group and let $K \subseteq G$ be a compact normal subgroup so that G/K is an R-group. Then G is an R-group.*

PROOF. Let X, μ be a standard probability G space with G acting freely. Since K is compact the space $\bar{X}$ of K orbits is a standard Borel G/K space (cf. [11] lemma 2) and the projection $p: X \to \bar{X}$ induces an invariant probability measure $\bar{\mu}$ on $\bar{X}$. Let λ be a left Haar measure on G and let $q: G \to G/K$ be projection. $q_*\lambda$ is a σ-finite left invariant measure on G/K, since K is compact. Let $E, F \subseteq G$ be compact and let $\delta, \varepsilon > 0$ be given. Choose a $q(F), \varepsilon$ invariant R-set $A \subseteq G/K$ with $q(E) \subseteq A$. Choose an A base $V \subseteq \bar{X}$ with $\bar{\mu}(AV) > 1-\delta$. Let $T: \bar{X} \to X$ be a Borel section of q. Then $T(V)$ is an AK base in $X, \mu(AKT(V)) > 1-\delta$ and AK is open, relatively compact, F, ε invariant, and $E \subseteq AK$.

PROPOSITION 1.10. *Let $G = \bigcup_{n=1}^{\infty} G_n$ where $G_1 \subseteq G_2 \subseteq \cdots$ are open subgroups of G which are R-groups. Then G is an R-group.*

PROOF. It is enough to observe that the Haar measure on G_n is the restriction of Haar measure on G, and that any compact set $J \subseteq G$ is contained in some G_n.

We are now in a position to apply the structure theory of l.c.s.c. groups to obtain the following:

COROLLARY 1.11. *Any l.c.s.c. abelian group is an R-group.*

PROOF. By 1.10 it is enough to consider compactly generated abelian groups. Any such group is of the form $K \times \mathbf{R}^n \times \mathbf{Z}^m$, where K is compact. The result follows by [8] and 1.9.

COROLLARY 1.12. *Any solvable l.c.s.c. group is an R-group.*

PROOF. First assume G connected. By Gleason's theorem there is a compact normal subgroup $K \subseteq G$ so that G/K is a solvable Lie group. By 1.9 it is sufficient to see G/K is an R-group. G/K is built up by a finite number of extensions of an abelian group by $\mathbf{R}$ or a compact group. Since all the groups are Lie groups these are all piecewise continuous extensions. The result follows by 1.7, 1.8 and 1.11.

If G is not connected, let G_0 be the identity component. G is an extension of G_0 by a discrete solvable group; hence we need only consider extensions of R-groups by discrete abelian groups. The result follows as in [14], or from 1.7, 1.8 and 1.10.

COROLLARY 1.13. *Almost connected amenable l.c.s.c. groups are R-groups.*

PROOF. G almost connected means G/G_0 compact. 1.8 reduces to the case of connected amenable groups, and Gleason's theorem and 1.9 reduces to connected amenable Lie groups. These are precisely those groups for which $G/\mathrm{rad}\, G$ is compact, where $\mathrm{rad}\, G$ is solvable. 1.12 and 1.8 give the result.

§2. Rohlin's Theorem for split extensions of R-groups

In this section we prove Theorem 1.7. Throughout, H is an l.c.s.c. R-group and G is a split extension of H by $\mathbf{R}$, $G = \mathbf{R} \circledS_\tau H$ where $\tau_t(h) = t^{-1}ht$. λ will denote a left Haar measure on H and h will be Lebesgue measure on $\mathbf{R}$. X will be a standard Borel G space, and G will act freely preserving a probability measure μ. We write $I(T) = [-T, T]$. $h \times \lambda$ is a left Haar measure on G.

REMARK 2.1. The map $\mathbf{R} \times H \to H$, $(t, h) \to \tau_t(h)$ is necessarily Borel and hence continuous by [12] proposition 1.4. The topology on G is therefore that of $\mathbf{R} \times H$.

REMARK 2.2. If H is discrete and $(t, h) \to \tau_t(h)$ continuous, then we must have $\tau_t(h) \equiv h$, so that $G = \mathbf{R} \times H$.

We will give our proof in the case of continuous H. At those points in our argument where we use the continuity of λ it will be seen that we could equally well use $\tau \equiv$ identity, so that we in fact prove the result in case H discrete also.

LEMMA 2.3. *There are arbitrarily large arbitrarily left invariant open relatively compact sets in G of the form $I(T)A$, where $A \subseteq H$ is an R-set.*

PROOF. Suppose compact sets $E, F \subseteq G$ and $\varepsilon > 0$ are given. Find compact sets $E_1, F_1 \subseteq H$ and $S, T > 0$ such that $E \subseteq I(S)E_1$, $F \subseteq I(T)F_1$. Pick $S_0 > S$ so that $T/S_0 < 1 - (1-\varepsilon)^{1/2}$, and choose $A \subseteq H$ to be an $F_2 = \bigcup_{|t| \leq S_0} \tau_t(F_1)$, $1 - (1-\varepsilon)^{1/2}$ invariant R-set containing E_1.

Let $B = \{a \in A : fa \in A \ \forall f \in F_2\}$. $\lambda(B) > (1-\varepsilon)^{1/2}\lambda(A)$. $I(T)F_1I(S_0 - T)B \subseteq I(S_0)A$. Moreover

$$h \times \lambda(I(S_0 - T)B) = 2(S_0 - T)\lambda(B) > 2S_0\lambda(A)(1-\varepsilon)^{1/2}(1 - T/S_0)$$
$$> 2S_0\lambda(A)(1-\varepsilon)$$
$$= (1-\varepsilon)h \times \lambda(I(S_0)A).$$

$F \subseteq I(T)F_1$ and $E \subseteq I(S_0)A$ so that $I(S_0)A$ is sufficiently invariant.

COROLLARY 2.4. *To prove Theorem* 1.7 *it is sufficient to show that sets of the form $I(T)A$ are R-sets in G whenever A is an R-set in H.*

Theorem 1.7 follows from the following two results:

PROPOSITION 2.5. *Let $A \subseteq H$ be an R-set and let $T > 0$, $\varepsilon > 0$ be given. Then there exist a relatively compact open set $J = J(A, T, \varepsilon) \subseteq H$, $S = S(A, T, \varepsilon) > 0$, and $\alpha = \alpha(A, T, \varepsilon) > 0$ such that if B is a J, α invariant R-set in H and $T' > S$, and if $\bar{V} \subseteq X$ is an $I(T')B$ tower, then there is an $I(T)A$ tower $\bar{W} \subseteq \bar{V}$ such that $\mu(\bar{W}) > (1-\varepsilon)\mu(\bar{V})$.*

PROPOSITION 2.6. *There exists $a > 0$ so that for any R-set $A \subseteq H$ and any $T > 0$, there are a symmetric open relatively compact set $F = F(A, T) \subseteq H$, and $U = U(A, T) > 0$, $\beta = \beta(A, T) > 0$ such that in any $I(U)F$, β invariant set $Z \in \mathcal{B}(X)$ with $\mu(Z) > 0$, an $I(T)A$ tower $\bar{W}$ may be found with $\mu(\bar{W}) > a\mu(Z)$.*

PROOF OF THEOREM 1.7. It is sufficient to show that given an R-set $A \subseteq H$ and $T > 0$, $\varepsilon > 0$ there is an $I(T)A$ tower $\bar{V}$ with $\mu(\bar{V}) > 1 - \varepsilon$.

Choose n so that $(1-a)^n < (1-\varepsilon)^{1/2}$. Set $\varepsilon_1 = 1 - (1-\varepsilon)^{1/2}$. Choose λ so that $x/(1-x) < \lambda$ for $0 \leqq x \leqq (1-\varepsilon)^{1/2}$. Inductively find R-sets $A_r \subseteq H$, and $T_r > 0$, $\phi_r > 0$, $r = 1, \cdots, n$, so that

(1) $A_1 = A$, $T_1 = T$, $\phi_1 = \alpha(A, T, \varepsilon_1)$,

(2) A_r is $J(A, T, \varepsilon_1)$, $\alpha(A, T, \varepsilon_1)$ invariant and $T_r > S(A, T, \varepsilon_1)$,

(3) $I(T_{r+1})A_{r+1}$ is $(I(U(A_r, T_r))F(A_r, T_r))^{-1}$, $(1/\lambda)\beta(A_r, T_r)$ invariant,

where we use the notation of 2.5, 2.6. The choice of (3) can be made using the method of Lemma 2.3.

By Proposition 2.6 find an $I(T_n)A_n$ tower H_n with $\mu(H_n) > a$. If $\mu(H_n) > (1-\varepsilon)^{1/2}$, stop. Otherwise $\mu(H_n)/\mu(X - H_n) < \lambda$ and from the $(I(U(A_{n-1}, T_{n-1}))F(A_{n-1}, T_{n-1}))^{-1}$, $\beta(A_{n-1}, T_{n-1})/\lambda$ invariance of $I(T_n)A_n$, we see that $X - H_n$ is $I(U(A_{n-1}, T_{n-1}))F(A_{n-1}, T_{n-1})$, $\beta(A_{n-1}, T_{n-1})$ invariant. Use 2.6 to choose an $I(T_{n-1})A_{n-1}$ tower $H_{n-1} \subseteq X - H_n$ with $\mu(H_{n-1}) > a\mu(X - H_n)$. $\mu(X - H_n - H_{n-1}) < (1-a)^2$. If $\mu(H_n \cup H_{n-1}) > (1-\varepsilon)^{1/2}$, stop, otherwise continue in this way. The process terminates on or before the nth step. We obtain disjoint sets $H_n, H_{n-1}, \cdots, H_k$ so that

(4) H_r is an $I(T_r)A_r$ tower,

(5) $\mu(\bigcup_k^n H_r) > (1-\varepsilon)^{1/2}$.

Use Proposition 2.5 to find $I(T)A$ towers V_r in each set H_r with $\mu(V_r) > \mu(H_r)(1-\varepsilon)^{1/2}$. Then $\bigcup_k^n V_r$ is an $I(T)A$ tower V with $\mu(V) > \mu(\bigcup_k^n H_r)(1-\varepsilon)^{1/2} > 1-\varepsilon$.

We turn to the proof of Proposition 2.5.

LEMMA 2.7. *Let $Q \subseteq H$ be an R-set and let $\varepsilon > 0$. Let $Z \in \mathscr{B}(X)$, $\mu(Z) > 0$, be QQ^{-1}, $\varepsilon/3$ invariant. Then there is a Q tower $\bar{V} \subseteq Z$ with $\mu(\bar{V}) > \mu(Z)(1-\varepsilon)$.*

PROOF. Let $Z' = \{z \in Z : hz \in Z\ \forall h \in QQ^{-1}\}$. $\mu(Z') > (1-\varepsilon/3)\mu(Z)$. Choose a Q tower $\bar{W} \subseteq X$ so that $\mu(Z - \bar{W}) < \frac{1}{3}\varepsilon\mu(Z)$. Choose α with $1-\varepsilon < \alpha < 1-2\varepsilon/3$. Let ν be the measure induced on the base W of $\bar{W}$ by $\bar{W}$. Suppose

$$\nu(Z' \cap qW) \leqq \frac{\alpha\mu(Z)}{\lambda(Q)} \quad \forall q \in Q.$$

Then

$$\mu(Z' \cap \bar{W}) \leqq \alpha\mu(Z),$$

so $\mu(Z)(1-2\varepsilon/3) < \mu(Z' \cap \bar{W}) < (1-2\varepsilon/3)\mu(Z)$. Hence $\exists q_0 \in Q$ such that $\nu(Z' \cap q_0 W) > (\alpha\mu(Z)/\lambda(Q)$, and $\nu(q_0^{-1}Z' \cap W) > \alpha\mu(Z)/\lambda(Q)$. $Q(q_0^{-1}Z' \cap W) \subseteq Z$ and $q_0^{-1}Z' \cap W$ is a Q base, also

$$\mu(Q(q_0^{-1}Z' \cap W)) = \int_Q \nu(q(q_0^{-1}Z' \cap W))d\lambda(q) > \alpha\mu(Z) > \mu(Z)(1-\varepsilon).$$

LEMMA 2.8. *Let $A \subseteq H$ be an R-set, and let $T > 0$, $\varepsilon > 0$, $\delta > 0$ with $(1+\delta)(1-\varepsilon) < 1$ be given. Let $\bar{V} \subseteq X$ be an $I(T(1+\delta))$ tower on a base V such that, with respect to the induced measure ω on V, the sets tV are AA^{-1}, ε' invariant whenever $|t| \leqq \delta T$, where $\varepsilon' = \frac{1}{4}\{1-(1+\delta)(1-\varepsilon)\}$. Then there is an $I(T)A$ tower $\bar{W} \subseteq \bar{V}$ with $\mu(\bar{W}) > \mu(\bar{V})(1-\varepsilon)$.*

PROOF. Set $\bar{U} = I(\delta T)V$, $V_t = tV$, $V'_t = \{x \in V_t : hx \in V_t\ \forall h \in AA^{-1}\}$, $\bar{U}' = \bigcup_{|t| \leqq \delta T} V'_t$. Then $\bar{U}'$ is measurable and

$$\omega(V'_t) \geqq (1-\varepsilon')\omega(V_t), \tag{1}$$

$$\mu(\bar{U}') \geqq (1-\varepsilon')\mu(\bar{U}). \tag{2}$$

By Lemma 2.7 there is an A tower $\bar{Y} \subseteq \bar{U}$ on a base Y with $\mu(\bar{Y}) > (1-3\varepsilon')\mu(\bar{U})$, and we see from the proof of 2.7 that we may suppose $Y \subseteq A^{-1}\bar{U}'$.

$\mu(\bar{U}'\cap\bar{Y})>(1-4\varepsilon')\mu(\bar{U})$ by (2). Therefore $\exists t_0\in\mathbf{R}$, $|t_0|\leqq\delta T$, such that

$$\omega(V'_{t_0}\cap\bar{Y})>(1-4\varepsilon')\omega(V). \tag{3}$$

Now $A(A^{-1}V'_{t_0}\cap Y)\subseteq V_{t_0}$ by the definition of V'_{t_0}. If $x\in V'_{t_0}\cap\bar{Y}$, then $x=ay$, $a\in A$, $y\in Y$. Moreover $y=a'^{-1}s$, $a'\in A$, $s\in\bar{U}'$; hence $s=a'a^{-1}x\in V_{t_0}$, and so $s\in V'_{t_0}$. Thus

$$V'_{t_0}\cap\bar{Y}\subseteq A(A^{-1}V'_{t_0}\cap Y)\subseteq V_{t_0}. \tag{4}$$

The sets $ta(A^{-1}V'_{t_0}\cap Y)$, $|t|\leqq T$, $a\in A$, are disjoint and

$$\begin{aligned}\mu(I(T)A(A^{-1}V'_{t_0}\cap Y)) &= 2T\omega(A(A^{-1}V'_{t_0}\cap Y))\\ &\geqq 2T(1-4\varepsilon')\omega(V) \qquad \text{by (3) and (4)}\\ &= (1-4\varepsilon')\frac{2T}{2T(1+\delta)}\mu(\bar{V})\\ &= (1-\varepsilon)\mu(\bar{V}).\end{aligned}$$

COROLLARY 2.9. *Let $A,T,\varepsilon,\delta,\varepsilon'$ be as above. Let $B\subseteq H$ be a $\bigcup_{|t|\leqq\delta T}T_t(AA^{-1})$, ε' invariant R-set in H. Then given any $I(T(1+\delta))B$ tower $\bar{V}$ there is an $I(T)A$ tower $\bar{W}\subseteq\bar{V}$ such that $\mu(\bar{W})>\mu(\bar{V})(1-\varepsilon)$.*

PROOF. Let V be the base of $\bar{V}$. It is easy to check that the sets tV, $|t|\leqq\delta T$, are AA^{-1}, ε' invariant with respect to the induced measure on V. Then apply the Lemma.

PROOF OF PROPOSITION 2.5. Let $A\subseteq H$ be an R set and let $T>0$, $\varepsilon>0$ be given. Choose ε' so that $(1-\varepsilon')^2>1-\varepsilon$, and $\delta>0$ so that $(1-\varepsilon')(1+\delta)<1$. Set $S=T(1+\delta)/\varepsilon'$, $J=\bigcup_{|t|\leqq\delta T}T_t(AA^{-1})$, $\alpha=\frac{1}{4}\{1-(1+\delta)(1-\varepsilon')\}$. Suppose $\bar{V}\subseteq X$ is an $I(T')B$ tower where $T'>S$ and B is a J, α invariant R-set in H. Divide $I(T')$ into disjoint intervals K_i of length $2T(1+\delta)$ so that the remaining part has length $<2T(1+\delta)<2\varepsilon'T'$. Each K_i determines an $I(T(1+\delta))B$ tower V_i. Apply Corollary 2.9 to find $I(T)A$ towers $W_i\subseteq V_i$ with $\mu(W_i)>\mu(V_i)(1-\varepsilon')$. $\mu(\bigcup W_i)>(1-\varepsilon')\mu(\bigcup V_i)>(1-\varepsilon')^2\mu(V)>(1-\varepsilon)\mu(V)$. $\bigcup W_i$ is the required $I(T)A$ tower.

LEMMA 2.10. *Let P, Q be probability spaces with measures p,q and suppose $E\in\mathscr{B}(P\times Q)$, $p\times q(E)>0$, and $\delta, 0<\delta<1$, are given. Then there is a set $N\in\mathscr{B}(Q)$ so that $q(N)>0$ and $\operatorname{ess\,sup}_{s\in M}q(E\cap\{s\}\times N)>\delta q(N)$.*

PROOF. Set $m=p\times q$ and choose $\varepsilon,\varepsilon'>0$ so that $\varepsilon'<1-\delta$ and $\varepsilon/(1-\varepsilon)<$

ε'. Find sets $P_i \in \mathcal{B}(P)$, $Q_i \in \mathcal{B}(Q)$, $i = 1, \cdots, n$, with $m(E \,\Delta\, \bigcup_{i=1}^n P_i \times Q_i) < \varepsilon m(E)$. By subdivision we may asume $\bigcup_{i=1}^n P_i \times Q_i$ to be a disjoint union of rectangles.

If $m(P_i \times Q_i - E) \geqq \varepsilon' m(P_i \times Q_i)$, $\forall i$ then

$$m\left(\bigcup_1^n P_i \times Q_i - E\right) \geqq \varepsilon' m\left(\bigcup_1^n P_i \times Q_i\right) \geqq \varepsilon'(1-\varepsilon) m(E)$$

and $\varepsilon m(E) \geqq \varepsilon'(1-\varepsilon)m(E)$ which is impossible.

Choose i_0 with $m(P_{i_0} \times Q_{i_0} - E) < \varepsilon' m(P_{i_0} \times Q_{i_0})$. Then

$$\text{(1)} \quad \int_{P_{i_0}} m(E \cap \{x\} \times Q_{i_0}) dp(x) = m(E \cap P_{i_0} \times Q_{i_0}) \geqq (1-\varepsilon') m(P_{i_0} \times Q_{i_0}).$$

$$\operatorname*{ess\,sup}_{s \in P_{i_0}} m(E \cap \{x\} \times Q_{i_0}) \leqq \delta q(Q_{i_0})$$

$$\Rightarrow \int_{P_{i_0}} m(E \cap \{x\} \times Q_{i_0}) dp(x) \leqq \delta m(P_{i_0} \times Q_{i_0}).$$

This contradicts (1), so the result is proved.

Lemma 2.11. *Let $Q \subseteq H$ be an R-set and let $V \subseteq X$ be a Q base with measure ν induced by QV. Let $T > 0$, $\delta > 0$, $\gamma > 0$, $\delta_1 > 0$ be given with $\delta < 1 - \gamma$, and let $P \subseteq Q$ be compact with $\lambda(P) > (1-\gamma)\lambda(Q)$. Then there are an $I(T)Q$ base $W \subseteq X$ and a set $J \subseteq I(\delta_1)$ and $V_1 \subseteq V$ so that*

(1) $\nu(V_1) > 0$, $h(J) > 0$,

(2) $PV_1 \subset JQW$,

(3) $\omega(tQW \cap QV_1) > \delta\omega(QW)$ $\forall t \in J$,

where ω is the measure induced on QW by the $I(T)$ tower $I(T)QW$.

Proof. Use compactness of P and continuity of τ to find a symmetric neighborhood U of $e \in H$, and $\varepsilon > 0$, so that $\tau_t(PU)U \subseteq Q$ for $|t| \leqq \varepsilon$; and so that

$$I(\varepsilon)U \subseteq I(T)Q \quad \text{and} \quad \varepsilon < \delta_1.$$

By Proposition 1.1, $\mu(UV) > 0$. Apply Proposition 1.3 to find an $I(T)Q$ base $W_1 \subset UV$ such that $\mu(I(\varepsilon)UW_1 \cap UV) > 0$. Let η be the measure induced on W_1 by the $I(\varepsilon)U$ tower $I(\varepsilon)UW_1$. Choose δ' with $\delta/(1-\gamma) < \delta' < 1$. By Lemma 2.10 there is a set $W \subseteq W_1$ so that $\eta(W) > 0$, and $h \times \lambda(F) > 0$, where $F = \{(t,u) \in I(\varepsilon) \times U\colon \eta(tuW \cap UV) > \delta'\eta(W)\}$. Let $\pi\colon QV \to V$ be projection and set $V_1 = \pi(FW \cap UV)$. Then $\nu(V_1) > 0$. Set $J = \{t \in \mathbf{R}\colon (t,u) \in F$ some $u \in U\}$. Certainly $h(J) > 0$ and $J \subseteq I(\delta_1)$. Moreover

$$s \in V_1 \Rightarrow us = tu_1w \text{ for some } (t, u_1) \in F,\ u \in U,\ w \in W$$
$$\Rightarrow ps = pu^{-1}tu_1w = t\tau_t(pu^{-1})u_1w \in JQW \text{ whenever } p \in P.$$

Hence $PV_1 \subseteq JQW$.

Also

$$(t,u) \in I(\varepsilon) \times U \Rightarrow \{w \in W : tuw \in UV_1\} \subseteq \{w \in W : tpw \in QV_1\} \forall p \in P,$$

because $tuw = u_1v_1$, $u_1 \in U$, $v_1 \in V_1 \Rightarrow tpw = \tau_{-t}(pu^{-1})u_1u_1^{-1}tuw = \tau_{-t}(pu^{-1})u_1v_1 \in QV_1$. Therefore

$$(t,u) \in F \Rightarrow \delta'\eta(W) \leqq \eta(tuW \cap UV) = \eta(tuW \cap UV_1)$$
$$\leqq \eta(tpW \cap QV_1) \forall p \in P$$
$$\Rightarrow \delta'\lambda \times \eta(PW) \leqq \lambda \times \eta(tPW \cap QV_1).$$

Since $\omega = \lambda \times \eta$,

$$\omega(tQW \cap QV_1) \geqq \omega(tPW \cap QV_1) \geqq \delta'\omega(PW) = \delta'\omega(QW)\frac{\lambda(P)}{\lambda(Q)} > \delta\omega(QW).$$

Remark 2.12. Notice that we may suppose U chosen so that $\mu(QU) < 2\mu(Q)$.

Proof of Proposition 2.6. Let $A \subseteq H$ be an R-set and let $T > 0$. We make the following choices:

(i) $\varepsilon_1 = 1 - 2^{-1/2}$;
(ii) $0 < \bar{\delta} < 1$, $(1 + \bar{\delta})(1 - \varepsilon_1) < 1$;
(iii) $\varepsilon_1' = \frac{1}{4}\{1 - (1 + \bar{\delta})(1 - \varepsilon_1)\}$;
(iv) $T' = T(1 + \bar{\delta})$;
(v) $0 < \delta < 1$, $1/\delta[1/50 + (1 - \delta)/\delta] < 1/16$;
(vi) $0 < \gamma < 1$, $\gamma < 1 - \delta$;
(vii) $T'' = T'(2 + \delta)$;
(viii) $\beta > 0$, $x - \beta > (1 - \varepsilon_1')(x + \beta)$ whenever $x \geqq \frac{1}{2}$;
(ix) $Q \subseteq H$ a $\bigcup_{|t| \leqq T(1+\bar{\delta})}^{|t'| \leqq T(1+2\bar{\delta})} \tau_1(AA^{-1})\tau_1(AA^{-1})$, β invariant R-set;
(x) $P \subseteq Q$ compact, $\lambda(P) > (1 - \gamma)\lambda(Q)$;
(xi) $U \subseteq H$ a symmetric neighborhood of $e \in H$, $\varepsilon > 0$ so that $\tau_t(PU)U \subseteq Q$ for $|t| \leqq \varepsilon$ and $\lambda(QU) < 2\lambda(Q)$;
(xii) β such that $(1 - 3\beta)(1 - \beta) > 2^{-1/2}$;
(xiii) $Q' = QU$.

We will show that $F(A, T) = (QU)(QU)^{-1}$, $U(A, T) = T''$, $\beta(A, T) = \beta$, $a =$

1/400 are suitable choices to ensure the conclusion of 2.6. For convenience the proof is divided into a sequence of steps. $Z \in \mathcal{B}(X)$ is an $I(U)F$, β invariant set with $\mu(Z) > 0$.

(A) We find a set $Z'' \subseteq Z$ so that all towers constructed in the proof on bases in Z'' lie inside Z.

Set

$$Z' = \{z \in Z : tz \in Z \ \forall |t| \leqq T''\},$$

$$Z'' = \{z \in Z : thz \in Z \ \forall |t| \leqq T'', \ h \in F\}.$$

Then $FZ'' \subseteq Z$ and $\mu(Z'') \geqq (1-\beta)\mu(Z)$. By Lemma 2.7 there is a Q tower $\bar{V} \subseteq Z'$ with $\mu(\bar{V}) \geqq (1-3\beta)\mu(Z') \geqq (1-3\beta)(1-\beta)\mu(Z) > \mu(Z)2^{-1/2}$ by (xii). The base V of $\bar{V}$ may be chosen with $V \subset Q'^{-1}Z''$, hence $Q'V \subseteq Z'$ and $I(T'')Q'V \subseteq Z$.

(B) We fill most of Z with $I(T'')Q$ towers of a special kind. Use conditions (vi), (x), (xi) and apply Lemma 2.11 repeatedly with an exhaustion argument to find disjoint sets $V_i \subseteq V$, and $I(T'')Q$ bases $W_i \subseteq UV_i$, and sets $J_i \subseteq I(\delta T')$ such that if ν is the measure induced on V by the tower $\bar{V}$, and ω_i the measure induced on QW_i by the tower $I(T'')QW_i$,

(1) $\nu(V_i) > 0$, $\varepsilon_i = h(J_i) > 0$, $\bigcup_{i=1}^{\infty} V_i = V$,

(2) $PV_i \subseteq J_iQW_i$,

(3) $\omega_i(tQW_i \cap QV_i) > \delta\omega(QW_i) \ \forall t \in J_i$.

By (A), $I(T'')QW_i \subseteq Z$. For $N \in \mathcal{B}(X)$, write $\bar{N} = QN$. For $S > 0$, set $E(S) = [0, S]$, $E^*(S) = (0, S]$. Since the sets W_i are disjoint and since $\omega_i = \omega_j$ on $t_iW_i \cap t_jW_j$ we may without ambiguity define ω to be a measure on sets of the form $\bigcup W_i$ or tW_i, where $\omega|_{tW_i} = \omega_i$, using 1.2 and unimodularity of $\mathbf{R}$.

(C) We inductively choose bases W_i whose $\mathbf{R}$ translates are sufficiently disjoint to build large $I(T')$ towers.

Suppose that $W_1, \cdots, W_m$ have been chosen such that

$$(4)_i \qquad 2\omega(L_i) > \omega(M_i), \qquad i = 1, \cdots, m$$

where $M_i = \bigcup_{j=1}^{i} \bar{W}_j$, $L_i = M_i - E^*(2T')M_i$.

Suppose that W_{m+1} satisfies

$$(5)_{m+1} \qquad 4\omega(\bar{W}_{m+1} \cap E(2T')M_m) < \omega(\bar{W}_{m+1}),$$

$$(6)_{m+1} \qquad 4\omega(\bar{W}_{m+1} \cap E(-2T')M_m) < \omega(\bar{W}_{m+1}).$$

We show that $(4)_{m+1}$ holds also:

$$L_{m+1} = (\bar{W}_{m+1} - E(2T')M_m) \cup (L_m - E(2T')\bar{W}_{m+1}),$$

(7) $$\omega(\bar{W}_{m+1} - E(2T')M_m) \geqq \tfrac{3}{4}\omega(\bar{W}_{m+1}), \qquad \text{by } (5)_{m+1}.$$

Also, by Proposition 1.2:

(8) $$\omega(L_m \cap E(2T')\bar{W}_{m+1}) = \omega(E(-2T')L_m \cap \bar{W}_{m+1}),$$

$$\omega(L_m - E(2T')\bar{W}_{m+1}) = \omega(L_m) - \omega(L_m \cap E(2T')\bar{W}_{m+1})$$

$$> \tfrac{1}{2}\omega(M_m) - \tfrac{1}{4}\omega(\bar{W}_{m+1})$$

by (8), $(4)_m$, $(6)_{m+1}$. Hence $2\omega(L_{m+1}) \geqq \omega(M_{m+1})$ by (7).

(D) We show that sets W_i may be chosen satisfying $(5)_i$, $(6)_i$ until $100\mu(E(2T')M_n) > \mu(\bar{V})$ or $100\mu(E(-2T')M_n) > \mu(\bar{V})$. Suppose W_i, $i = 1, \cdots, m$ satisfy $(5)_i$, $(6)_i$ and $100\mu(E(2T')M_m) < \mu(\bar{V})$. Write $M = M_m$. Set

$$f(x) = \sum_{i=1}^{\infty} \omega(\bar{W}_i \cap E(2T')M)\chi_{\bar{V}_i}(x)\omega(\bar{W}_i)^{-1}, \quad x \in \bar{V}.$$

We estimate $\int_{\bar{V}} f(x)d\mu(x)$:

$$\mu(\bar{V}_i) = \mu(PV_i)\lambda(Q)\lambda(P)^{-1} < (1-\gamma)^{-1}\mu(J_i\bar{W}_i) = (1-\gamma)^{-1}\varepsilon_i\omega(\bar{W}_i)$$

by (x), (1) and (2);

$$\mu(J_i\bar{W}_i \cap \bar{V}_i) > \delta\varepsilon_i\omega(\bar{W}_i)$$

by (1), (3). Hence

(9) $$\delta\varepsilon_i\omega(\bar{W}_i) < \mu(\bar{V}_i) < \varepsilon_i\omega(\bar{W}_i)(1-\gamma)^{-1}.$$

Write $K = E(T'')$. $J_iE(2T') \subseteq K$, $\forall i$ by choice of J_i.

(10) $$\omega(E(2T')M \cap \bar{W}_i) = \omega(tE(2T')M \cap t\bar{W}_i) \leqq \omega(KM \cap t\bar{W}_i), \quad t \in J_i,$$

(11) $$\mu(\bar{V}_i \cap KM) \geqq \mu(\bar{V}_i \cap KM \cap J_i\bar{W}_i) = \int_{J_i} \omega(\bar{V}_i \cap KM \cap t\bar{W}_i)dt$$

$$= \int_{J_i} \omega(KM \cap t\bar{W}_i)dt - \int_{J_i} \omega(KM \cap (t\bar{W}_i - \bar{V}_i))dt$$

$$\geqq \varepsilon_i\omega(E(2T')M \cap \bar{W}_i) - \varepsilon_i(1-\delta)\omega(\bar{W}_i)$$

by (10) and (3).

$$\int_{\bar V} f(x)d\mu(x) = \sum_{i=1}^{\infty} \omega(\bar W_i \cap E(2T')M)\mu(\bar V_i)\omega(\bar W_i)^{-1}$$

$$\leqq \sum_{i=1}^{\infty} \omega(\bar W_i \cap E(2T')M)\varepsilon_i(1-\gamma)^{-1} \qquad \text{by (9)}$$

$$\leqq (1-\gamma)^{-1}\left[\sum_{i=1}^{\infty} \mu(\bar V_i \cap KM) + \sum_{i=1}^{\infty} \varepsilon_i\omega(\bar W_i)(1-\delta)\right] \qquad \text{by (11)}$$

$$\leqq (1-\gamma)^{-1}[\mu(E(2T')M) + T'\delta\omega(M) + \delta^{-1}(1-\delta)\mu(\bar V)] \qquad \text{by (9)}$$

since $\mu([a,b]M) \leqq \Sigma_{i=1}^{n} \mu([a,b]\bar W_i) = |a-b|\Sigma_{i=1}^{n}\omega(\bar W_i) = |a-b|\omega(M)$.

Now $\omega(M_m) < 2\omega(L_m)$ by $(4)_m$, so $\delta T'\omega(M) < \delta\mu(E(2T')M)$,

$$\int_{\bar V} f(x)d\mu(x) \leqq (1-\gamma)^{-1}[\mu(E(2T')M)(1+\delta) + \delta^{-1}(1-\delta)\mu(\bar V)]$$

$$\leqq \delta^{-1}[\tfrac{2}{100} + \delta^{-1}(1-\delta)]\mu(\bar V) \qquad \text{by (vi)}$$

$$< \mu(\bar V)/16 \qquad \text{by (v).}$$

By Tchebychev's inequality, $\mu(\{x \in \bar V : f(x) \leqq \frac14\}) \geqq \frac34\mu(\bar V)$. Define

$$g(x) = \sum_{i=1}^{\infty} \omega(\bar W_i \cap E(-2T')M)\chi_{\bar V_i}(x)\omega(\bar W_i)^{-1}, \qquad x \in \bar V.$$

By exactly similar estimates we obtain $\mu(\{x \in \bar V : g(x) \leqq \frac14\}) \geqq \frac34\mu(\bar V)$. Therefore $f(x)$, $g(x)$ are simultaneously $\leqq \frac14$ on a set of positive measure, so that V_{m+1}, W_{m+1} satisfying $(5)_{m+1}$, $(6)_{m+1}$ may be chosen.

(E) Use (D) and an exhaustion argument to obtain $W_1, \cdots, W_n$ satisfying $(5)_i$, $(6)_i$ such that $\mu(E(2T')M_n) \geqq \mu(\bar V)/100$. (The argument for $E(-2T')$ is similar.)

By $(4)_n$, $2\omega(L_n) > \omega(M_n)$. Hence

$$\mu(E(2T')L_n) = 2T'\omega(L_n) > T'\omega(M_n) \geqq \tfrac12\mu(E(2T')M_n) \geqq \frac{\mu(\bar V)}{200} \tag{12}$$

$N = T'L_n$ is an $I(T')$ base. In (F) below we show that large subsets $tN' \subseteq tN$, $|t| \leqq \bar\delta T$, are AA^{-1}, ε_1' invariant. By Lemma 2.8 and (ii), (iii), (iv) there is an $I(T)A$ tower $\bar W \subseteq I(T')N'$ with $\mu(\bar W) > \mu(I(T')N')(1-\varepsilon_1)$. Then

$$\mu(\bar W) > \frac{(1-\varepsilon_1)\mu(\bar V)}{200} \qquad \text{by (12)}$$

$$> \frac{\mu(Z)}{400} \qquad \text{by (A) and (i).}$$

(F) Invariance of the sets $tN, |t| \leqq \bar{\delta}T$:
Let

$$Q_1 = \{q \in Q : \tau_t(AA^{-1})q \subseteq Q, \forall |t| \leqq T(1+2\bar{\delta})\},$$

$$Q_2 = \{q \in Q : \tau_{t'}(AA^{-1})\tau_t(AA^{-1})q \subseteq Q, \forall |t'| \leqq T(1+2\bar{\delta}), |t| \leqq T(1+\bar{\delta})\}.$$

Then $\tau_t(AA^{-1})Q_2 \subseteq Q_1$ for $|t| \leqq T(1+\bar{\delta})$ and $\lambda(Q_2) > (1-\beta)\lambda(Q_1)$ by (ix). Set

$$M = \bigcup_{i=1}^{n} \bar{W}_i, \quad M_i = \bigcup_{j=1}^{n} Q_i W_j, \qquad i = 1, 2;$$

$$L = M - E^*(2T)M, \quad L_i = M_i - E^*(2T)M_i, \qquad i = 1, 2.$$

$$AA^{-1}(t(M_2 - E^*(2T)M)) \subseteq tL_1 \tag{13}$$

whenever $|t - T'| \leqq \bar{\delta}T$ by choice of Q_1 and Q_2. By Proposition 1.1, $\mu\,|_{I(T)\bar{W}_i} = h \times \lambda \times \eta_i$ where η_i is the induced measure on W_i. Therefore $\omega_i = \lambda \times \eta_i$,

$$\begin{aligned} \omega(M_2 - E^*(2T)M) &= \omega(L) - \omega\left((Q - Q_2) \bigcup_{i=1}^{n} W_i \cap L\right) \\ &\geqq \omega(L) - \omega\left((Q - Q_2) \bigcup_{i=1}^{n} W_i\right) \\ &\geqq \omega(L) - \beta\omega(M). \end{aligned} \tag{14}$$

Since L_1 is an $E(2T)$ base, Proposition 1.2 shows that

$$\begin{aligned} \omega(E^*(2T)(Q - Q_1)W_i \cap L_1) &= \omega((Q - Q_1)W_i \cap E^*(-2T)L_1) \\ &\leqq \omega((Q - Q_1)W_i), \end{aligned}$$

$$\begin{aligned} \omega\left(E^*(2T)(Q - Q_1) \bigcup_{1}^{n} W_i \cap L_1\right) &\leqq \sum_{i=1}^{n} \omega((Q - Q_1)W_i) \\ &\leqq \beta \sum_{i=1}^{n} \omega(\bar{W}_i) = \beta\omega(M), \end{aligned}$$

$$\begin{aligned} \omega(L_1) &= \omega(M_1 - E^*(2T)M) + \omega\left(E^*(2T)(Q - Q_1), \bigcup_{i=1}^{n} W_i \cap L_1\right) \\ &\leqq \omega(L) + \beta\omega(M). \end{aligned} \tag{15}$$

By (14) and (15),

$$\frac{\omega(M_2 - E^*(2T)M)}{\omega(L_1)} \geqq \frac{\omega(L) - \beta\omega(M)}{\omega(L) + \beta\omega(M)} \geqq 1 - \varepsilon_1' \qquad \text{by (viii) and } (4)_n.$$

This with (13) gives the invariance of the sets $tL_1 = tN'$ which are close in size to tN.

§3. Piecewise continuous extensions of R-groups

We show how to modify §2 to prove Theorem 1.8. α will be a fixed piecewise continuous section $\alpha: L \to G$, and if $E \subseteq L$, $A \subseteq H$ we write $EA = \{\alpha(e)a: a \in E, a \in A\}$. Using α to identify G with $L \times H$ we see that $\nu = h \times \lambda$ is a left Haar measure on G. $\tau_p(h) = \alpha(p)^{-1}h\alpha(p)$, $p \in L$. If H is discrete τ is the identity in a neighborhood of e and this replaces the assumption that λ is continuous. The analogue of Lemma 2.1 will be Lemma 2.1*, etc.

Case 1: *$L = K$, compact*

LEMMA 2.3*. *To prove that G is an R-group it is sufficient to show that KA is an R-set whenever $A \subseteq H$ is an R-set.*

PROOF. Suppose $E, F \subseteq G$ are compact and $\varepsilon > 0$ is given. Let $E_k = E \cap kH$. $b': d \to \alpha(p(d))^{-1}d$ is continuous on $p^{-1}(K_i) \cap E$, where $p: G \to K$ is projection. Therefore $\Phi(E) = \bigcup_{k \in K} k^{-1}E_k$ is relatively compact.

Let $F_k = F \cap kH$, $\beta(k, k') = \alpha(k'k)\alpha(k)^{-1}$. Then $F_{k'} = \beta(k, k')G(k, k')$ where $G(k, k') \subseteq H$. $\Psi: f, k \mapsto \tau(k)(\alpha(k)\alpha(p(f)k)^{-1})f$ is continuous on $F \cap p^{-1}(K_i) \times K_j$ and hence $\Psi(F \times K) = \bigcup_{k, k' \in K} \tau(k)G(k, k')$ is relatively compact. Choose an R-set $B \subseteq H$ so that $\Phi(E) \subseteq B$ and B is $\Psi(F \times K)$, ε invariant. Then $KB \supseteq E$ and KB is open and relatively compact.

$$
\begin{aligned}
\nu(\{x \in KB: FX \not\subseteq KB\}) &= \int_K \lambda(\{x \in kB: Fx \not\subseteq KB\})dk \\
&= \int_K \lambda(\{x \in B: \tau(k)G(k, k')x \not\in B\ \forall k' \in K)dk \\
&< \varepsilon \int_K \lambda(B)dk \\
&= \varepsilon\nu(KB).
\end{aligned}
$$

Sets KB can therefore be chosen arbitrarily large and arbitrarily left invariant.

LEMMA 2.8*. *Let $A \subseteq H$ be an R-set and let $\varepsilon > 0$ be given. Let $\bar{V} \subseteq X$ be a K tower on a base V such that, with respect to the induced measure ω on V, the sets kV are all AA^{-1}, $\varepsilon/4$ invariant. Then there is a KA tower $\bar{W} \subseteq \bar{V}$ with $\mu(\bar{W}) > \mu(\bar{V})(1 - \varepsilon)$.*

REMARK. The induced measure ω on V has only been defined when V is of the form BW, $B \subseteq H$, and W is a KB tower. This condition is however always fulfilled when we apply the Lemma or make other use of the induced measure. Moreover it is not hard to see that one can modify the proof of Proposition 1.1 to show that the measure on a K tower is $h \times \nu$ where h is left Haar measure on K and ν is the induced measure. Notice that since K is unimodular, we have $T_*\eta = \nu$ in applications of Proposition 1.2.

PROOF. The modifications to make in 2.8 are clear.

Lemma 2.8* is a substitute for Proposition 2.5.

PROPOSITION 2.6*. *There exists $a > 0$ so that for any R-set $A \subseteq H$ there is a symmetric open relatively compact set $F = F(A) \subseteq H$ and $\beta = \beta(A) > 0$ such that in any KF, β invariant set $Z \in \mathcal{B}(X)$ with $\mu(Z) > 0$, a KA tower $\bar{W}$ may be found with $\mu(\bar{W}) > a\mu(Z)$.*

It is obvious how to modify the proof of 1.7 to obtain 1.8 from 2.6* and 2.8*. It remains to prove 2.6*.

$A \subseteq H$ is an R-set. Write $D = \{\alpha(k)\alpha(k')^{-1}: k, k' \in K\}$. Let $Y \subseteq H$ be compact, symmetric, and let $D \subseteq KY$ and $K^2 \subseteq KY$.

Make the following choices:

(i) $\varepsilon_1 = 1 - 2^{-\frac{1}{2}}$,

(ii) $0 < \delta < 1$, $\delta^{-1}(3/50 + (1-\delta)/\delta) < 1/16$,

(iii) $0 < \beta < 1/3$, $(2+\beta)/(1-\beta) < 6$, $(1-\beta)(1-3\beta) > 2^{-\frac{1}{2}}$, $(1-\beta)(1-\varepsilon_1) > \frac{1}{2}$, $(x - \beta) > (1 - \varepsilon_1/4)(x + \beta)$ for $x \geqq \frac{1}{2}$,

(iv) $Q' \leqq H$ a Y^2B, $1 - (1 + \beta/2)^{-1}$ invariant R-set where

$$B = \bigcup\{\tau_{k'}(\tau_p(AA^{-1}))Y: k' \in K, p \in \alpha(K) \cup \alpha(K)^{-1}\},$$

(v) $Q = \{q \in Q': Y^2q \subseteq Q'\}$,

(vi) $0 < \gamma < 1$, $1 - \delta > \gamma$,

(vii) $P \subseteq Q$ compact, $\lambda(P) > (1-\gamma)\lambda(Q)$,

(viii) $E \subseteq H$, $C \subseteq K$ symmetric neighborhoods of the identity with $\tau_k(PE)E \subseteq Q$ $\forall k \in C$, and $QE \subseteq Y^2Q \subseteq Q'$.

We will show that $F(A) = Q'Q'^{-1}$, $\beta(A) = \beta$, $a = 1/600$ satisfy the requirements. We begin by showing that Q is a B, β invariant R-set.

$\{q \in Q': Y^2Bq \subseteq Q'\} \subseteq \{q \in Q': Bq \subseteq Q\}$, and so

$$\mu(\{q \in Q': Bq \subseteq Q\}) > (1 - \beta/2)\mu(Q') \qquad \text{by (iv)}$$

$$\mu(\{q \in Q: Bq \subseteq Q\}) \geqq (1-\beta/2)\mu(Q') - \mu(Q' - Q)$$

$$\geqq (1-\beta)\mu(Q) \qquad \text{by (iv).}$$

Since Q towers almost as much of the space as Q' towers, we now need only see Q is open. So suppose $q_n \notin Q$, $q_n \to h \in H$. If $h \notin Q'$ then $h \notin Q$. Hence we may assume $q_n \in Q'$, since Q' is open. Then $\exists x_n \in Y^2$ so that $x_n q_n \notin Q'$. Since Y^2 is compact we may find a convergent subsequence $x_{n_i} \to x \in Y^2$. Then $x_{n_i} q_{n_i} \to xh$ and $xh \notin Q'$ since Q' is open. Therefore $h \notin Q$, so Q is open.

(A)* Proceed as in (A) to find a Q tower $\bar{V}$ with $\mu(\bar{V}) > \mu(Z)2^{-\frac{1}{2}}$ and $KQ'V \subseteq Z$.

(B)* The proof of Lemma 2.11 is almost unchanged. $J \subseteq C$ is no longer restricted; however we now require W to be a KQ' base. Choose W_i, V_i, J_i satisfying (1), (2), (3) of (B). By (A)*, $KQW_i \subseteq Z$. Set $D^* = D - \{e\}$, $\bar{W}_i = QW_i$, $\tilde{W}_i = Q'W_i$.

(C)* Suppose $W_1, \cdots, W_m$ have been chosen such that

$$(4)_i \qquad 2\omega(L_i) > (1-\beta)\omega(M_i), \qquad i = 1, \cdots, m$$

where $M_i = \bigcup_{j=1}^{i} \bar{W}_j$, $L_i = M_i - D^*M_i$. Notice that the translates of L_i by K are disjoint.

Suppose that W_{m+1} satisfies

$$(5)_{m+1} \qquad 4\omega(\bar{W}_{m+1} \cap D^*M_m) < \omega(\bar{W}_{m+1}).$$

Then $(4)_{m+1}$ holds also. The estimates are those of (C) except

$$
\begin{aligned}
(8)^* \qquad \omega(L_m \cap D^*\bar{W}_{m+1}) &\leqq \omega(\tilde{W}_{m+1} \in K^{-1}L_m) \qquad \text{by Proposition 1.2}\\
&\leqq \omega(\tilde{W}_{m+1} \cap D^*M_m)\\
&\leqq \omega(\bar{W}_{m+1} \cap D^*M_m) + \omega(\tilde{W}_{m+1} - \bar{W}_{m+1})\\
&< \tfrac{1}{4}\omega(\bar{W}_{m+1}) + \tfrac{1}{2}\beta\omega(\bar{W}_{m+1}).
\end{aligned}
$$

(D)* W_i may be chosen satisfying $(5)_i$ until $\mu(KM_n) \geqq 1/100\,\mu(\bar{V})$. Suppose $W_i, i = 1, \cdots, m$, satisfy $(5)_i$ and $100\,\mu(KM_m) < \mu(\bar{V})$. Write $M = M_m$. Set

$$f(x) = \sum_{i=1}^{\infty} (\bar{W}_i \cap DM)\chi_{\bar{V}_i}(x)\omega(\bar{W}_i)^{-1}, \qquad x \in \bar{V}.$$

Then 2.6 (9) follows as before.

$$
\begin{aligned}
(10)^* \qquad \omega(DM \cap \bar{W}_i) &\leqq \omega(KYM \cap \bar{W}_i)\\
&= \omega(kKYM \cap k\bar{W}_i) \leqq \omega(KY^2M \cap k\bar{W}_i),
\end{aligned}
$$

$$(11)^* \qquad \mu(\bar V_i \cap KY^2M) \geqq \mu(\bar V_i \cap KY^2M \cap J_i\bar W_i)$$

$$= \int_{J_i} \omega(\bar V_i \cap KY^2M \cap k\bar W_i)dk$$

$$= \int_{J_i} \omega(KY^2M \cap k\bar W_i)dk$$

$$- \int_{J_i} \omega(KY^2M \cap (k\bar W_i - \bar V_i))dk$$

$$\geqq \varepsilon_i\omega(DM \cap \bar W_i)$$

$$- (1-\delta)\varepsilon_i\omega(\bar W_i) \quad \text{by (3) and (10)*;}$$

$$\int_{\bar V} f(x)d\mu(x) = \sum_{i=1}^{\infty} \omega(\bar W_i \cap DM)\mu(\bar V_i)\omega(\bar W_i)^{-1}$$

$$\leqq \sum_{i=1}^{\infty} \omega(\bar W_i \cap DM)\varepsilon_i(1-\gamma)^{-1} \qquad \text{by (9)}$$

$$\leqq \sum_{i=1}^{\infty} (1-\gamma)^{-1}[\mu(\bar V_i \cap KY^2M) + \varepsilon_i(1-\delta)\omega(\bar W_i)] \qquad \text{by (11)*}$$

$$\leqq (1-\gamma)^{-1}[\mu(KY^2M) + (1-\delta)/\delta\mu(\bar V)] \qquad \text{by (9);}$$

$$\mu(KY^2M) \leqq \sum_{i=1}^{m} \mu(KY^2\bar W_j)$$

$$\leqq \sum_{j=1}^{m} \mu(KQ'W_j) \leqq \sum_{j=1}^{m} (1+\tfrac{1}{2}\beta)\omega(\bar W_j) \qquad \text{by (iv)}$$

$$= (1+\beta/2)\omega(M) \leqq (2+\beta)/(1-\beta)\sum_{j=1}^{m}\omega(L_j) \qquad \text{by (C)*}$$

$$= (2+\beta)/(1-\beta)\mu(KL) \leqq (2+\beta)/(1-\beta)\mu(KM).$$

Hence

$$\int_{\bar V} f(x)d\mu(x) \leqq 1/(1-\gamma)[(2+\beta)/(1-\beta)100^{-1} + (1-\delta)/\delta]\mu(\bar V)$$

$$< \mu(\bar V)/16 \qquad \text{by (ii), (iii) and (vi).}$$

We conclude as in (D) that we may choose V_{m+1}, W_{m+1} satisfying $(5)_{m+1}$.

(E)* By estimates similar to those of (E), using $(4)_n$, (C) and (iii) we obtain

$$\mu(KL_n) \geqq \mu(\bar V)/300.$$

In (F)* we find $L_1 \subseteq L_n$ so that $\omega(L_1) > (1-\beta)\omega(L_n)$ and so that the sets kL_1,

$k \in K$, are $AA^{-1}, \varepsilon_1/4$ invariant. By 2.8* there is a KA tower $\bar{W} \subseteq KL_1$ with $\mu(\bar{W}) > \mu(Z)/600$.

(F)* Let

$$Q_1 = \{q \in Q : \tau_p(AA^{-1})q \subseteq Q, \forall p \in \alpha(K) \cup \alpha(K)^{-1}\},$$

$$Q_2 = \{q \in Q : \tau_{k'}(\tau_p(AA^{-1}))Yq \subseteq Q, \forall k' \in K, p \in \alpha(K) \cup \alpha(K)^{-1}\}.$$

Then

$$\lambda(Q_2) > (1 - \beta/2)\lambda(Q_1) \qquad \text{and}$$

$$\lambda(YQ - Q_2) \leqq (Q' - Q_2) \leqq \beta\lambda(Q).$$

Set

$$M = \bigcup_{i=1}^{n} \bar{W}_i, \quad M_i = \bigcup_{j=1}^{n} Q_i W_j, \qquad i = 1, 2,$$

$$L = M - D^*M, \quad L_i = M_i - D^*M_i, \qquad i = 1, 2.$$

As in (F):

$$AA^{-1}(k(M_2 - D^*M)) \subseteq kL_1, \qquad \forall k \in K, \tag{13)*$$

$$\omega(M_2 - D^*M) \geqq \omega(L) - \beta\omega(M), \tag{14)*$$

$$\begin{aligned} \omega(L_1) &= \omega(M_1 - D^*M) + \omega\left(D^*(Q - Q_1) \bigcup_{i=1}^{n} W_i \cap L_1\right) \\ &\leqq \omega(L) + \omega\left(Y(Q - Q_1) \bigcup_{i=1}^{n} W_i\right) \\ &\leqq \omega(L) + \omega\left((YQ - Q_2) \bigcup_{i=1}^{n} W_i\right) \\ &\leqq \omega(L) + \beta\omega(M). \end{aligned} \tag{15)*$$

Hence $\omega(M_2 - D^*M)/\omega(L_1) \geqq 1 - \varepsilon_1/4$ by (14)*, (15)* and (iii).

Case 2: $L = \mathbf{R}$.

It should be clear by now that we can combine the methods of §2 and those of Case 1 above to prove Theorem 1.8. We note briefly the points at which the argument is modified.

LEMMA 2.3**. *This is similar to Lemma 2.3*. We work with an interval $K \subseteq \mathbf{R}$ which contains $p(F) \cup p(E)$ and assume K is a union of a finite number of sets on which α is continuous.*

PROPOSITION 2.5**. *With the notation of* 2.5, K_i *no longer determines an* $I(T(1+\delta))B$ *tower. Instead we replace* B *by a larger set* B' *which is chosen sufficiently invariant under a large set in* H *to ensure that* $\{\alpha(S_i)x : x \in V\}$ *form an* $I(T)B$ *tower which fills most of the space* $\bar{V}$, *where* S_i *is the left endpoint of* K_i.

The proofs of 1.7, 2.7, 2.8, 2.10, 2.11 are unchanged. The remark following 2.8* still applies.

PROPOSITION 2.6**. *In* (C) *we replace the set* $W_{m+1} \cap E(2T)M_m$ *by* $W_{m+1} \cap E(2T)YM_m$, $Y \subseteq H$ *being chosen to ensure* $M - E(2T)YM$ *has disjoint translates under* $E(2T)$, *and obtain estimates as in the compact case. The estimates of* (D) *are as in Case* 1 *and we treat* $E(-2T)$ *similarly. In* (E) *we must ensure* $T'L_n$ *is an* $I(T')$ *base and this may be done by requiring* Q *to be sufficiently invariant. The estimates of* (F) *then follow as in Case* 1.

§4. Hyperfiniteness of group actions

A measure preserving action of a group G on a probability space X, μ generates a natural equivalence relation on X. An equivalence relation on X is said to be countable (finite) if there are at most countably (finitely) many points in each orbit, and hyperfinite if it is an increasing union of finite relations (cf. e.g. [6]). It is known that all free measure class preserving actions of discrete solvable groups generate hyperfinite equivalence relations [1]. To generalize these ideas for continuous groups and uncountable relations we make the following definitions (see also [4]): An equivalence relation R is countably hyperfinite if it is the union of an increasing sequence of finite relations. Let $J_n = \{1, \cdots, n\}$, $n \in \mathbf{N}$, $J_0 = [0, 1]$. A relation R on X, μ is cyclic if

$$(1) \qquad X \cong \bigcup_{i=0}^{\infty} Y_n \times J_n, \qquad \text{where } Y_n \in \mathcal{B}(X),$$

the union is disjoint, and the isomorphism is measure theoretic in the sense that there is a measure λ_n on Y_n so that $\lambda_n \times h_n = \mu\,|_{Y_n \times J_n}$ where h_n is Lebesgue measure on J_n, and if

$$(2) \qquad x \sim y \Leftrightarrow x = (y_n, j_n), \quad y = (y_n, j'_n), \qquad \text{where } y_n \in Y_n \text{ and } j_n, j'_n \in J_n.$$

A relation R is hyperfinite if

(1) $\exists E \in \mathcal{B}(X)$ such that the saturation of E is conull in X and $R\,|_E$ is countable,

(2) R is the union of an increasing sequence of cyclic relations.

PROPOSITION 4.1. *Let* R *be a hyperfinite relation on a measure space* X, μ *and* *let $E \in \mathscr{B}(X)$ have conull saturation such that $R|_E$ is countable. Then $R|_E$ is countably hyperfinite. Conversely, if $\exists E \in \mathscr{B}(X)$ with conull saturation such that $R|_E$ is countably hyperfinite then R is hyperfinite.*

PROOF. The details are fairly straightforward. The result is in [4].

THEOREM 4.2. *Let G be a l.c.s.c. R-group, and X be a standard Borel G space on which G acts freely preserving a probability measure μ. Then the relation R generated by the G action is hyperfinite.*

PROOF. By [4], $\exists E \in \mathscr{B}(X)$ with conull saturation and so that $R|_E$ is countable. Let $B_0 = \{e\} \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be a sequence of compact sets with $\bigcup_{i=1}^{\infty} B_i = G$. Suppose inductively we have relations R_i, $i = 0, \cdots, n$ on X, and compact $A_i \subseteq G$, such that

(1) $R_n \supseteq R_{n-1} \supseteq \cdots \supseteq R_0$ where R_0 is the trivial relation,

(2) R_i is cyclic, $i = 1, \cdots, n$,

(3) $\mu(\{x \in X : A_{i-1}A_{i-1}^{-1}x \not\subseteq R_i x\}) < 2^{-i}$, $i = 1, \cdots, n$,

(4) $x \sim_i y \Rightarrow x = hy$, $h \in A_i A_i^{-1}$, $i = 1, \cdots, n$,

(5) $B_i \subseteq A_i$, $i = 1, \cdots, n$.

These conditions certainly hold for $R_0, B_0, A_0 = \{e\}$. We construct R_{n+1}, A_{n+1}. Choose A_{n+1} to be an $A_n A_n^{-1}, 2^{-(n+2)}$ invariant R-set with $A_n \cup B_{n+1} \subseteq A_{n+1}$. Find an A_{n+1} tower $\bar{E} \subseteq X$ on a base E such that $\mu(\bar{E}) > 1 - 2^{-(n+2)}$. Set

$$C = \{g \in A_{n+1} : hg \in A_{n+1} \forall h \in A_n A_n^{-1}\}.$$

Let Y be the saturation of CE under R_n. If $y \in Y$, then $y \sim_n gx$ where $g \in C$, $x \in E$. By (4), $y = hgx$, $h \in A_n A_n^{-1}$. By choice of C, $y \in \bar{E}$. Therefore $Y \subseteq \bar{E}$. Define R_{n+1} on Y to be the relation $x \sim_{n+1} y \Leftrightarrow x = ax_0$, $y = bx_0$ where $a, b \in A_{n+1}$, $x_0 \in E$. Define R_{n+1} on $X - Y$ to be the relation R_n. R_{n+1} is clearly cyclic by Proposition 1.1. Suppose $x, y \in Y$ and $x \sim_n y$. Then $x = az$, $y = bz$, $z \in CE$, $a, b \in A_n A_n^{-1}$. $z = gz_0$ where $g \in C$ and $z_0 \in E$. Thus $x = agz_0$, $y = bgz_0$, and $ag, bg \in A_{n+1}$, so $x \sim_{n+1} y$. Hence $R_n \subseteq R_{n+1}$.

Suppose $x \sim_{n+1} y$. If $x, y \in X - Y$ then $x \sim_n y$ and so $x = hy$, $h \in A_n A_n^{-1} \subseteq A_{n+1}A_{n+1}^{-1}$. If $x, y \in Y$ then $x = az$, $y = bz$ where $z \in E$, $a, b \in A_{n+1}$. Then $x = ab^{-1}y$.

$$x \in CE \Rightarrow A_n A_n^{-1} x \subseteq R_{n+1}x.$$

Hence

$$\mu(\{x \in X : A_n A_n^{-1}x \not\subseteq R_{n+1}x\}) \leqq \mu(X - CE) \leqq 2^{-(n+2)} + 2^{-(n+2)} = 2^{-(n+1)}.$$

R_{n+1}, A_{n+1} now satisfy (1)–(5). It is clear that $R = \bigcup_{i=1}^{\infty} R_i$, so that R is hyperfinite.

Any countably hyperfinite equivalence relation arises from a $\mathbf{Z}$ action, by [6] theorem 4.1. The corresponding result in the uncountable case is

PROPOSITION 4.3. *Let R be a hyperfinite equivalence relation on a space X, μ, such that there are uncountably many points in each orbit. Then R is generated by a flow.*

PROOF. Choose $E \in \mathscr{B}(X)$ with conull saturation so that $R\,|_E$ is countable and hence hyperfinite by Proposition 3.1. Let $p: X \to E$ be any measurable map such that $p(x) \sim x$ a.a. $x \in X$. Let $E_0 = \{e \in E: p^{-1}(e) \cong [0,1]\}$ ($\cong$ meaning a measure theoretic isomorphism of $[0,1]$ with Lebesgue measure and $p^{-1}(e)$ and the induced fibre measure). Clearly $p_*(\mu)(E_0) > 0$. Find also a map $q: p^{-1}(E - E_0) \to E_0$ preserving R. It is clear that $X \cong I \times E_0$ (cf. e.g. [15] appendix) and that $(t,x) \sim (s,x)$ $\forall t, s \in I$, $x \in E_0$. Find a transformation T generating $R\,|_{E_0}$. The flow built on $E_0, p_*\mu$, T under the constant function 1 generates R.

REFERENCES

1. A. Connes and W Krieger, *Measure space automorphisms, the normalizers of their full groups, and approximate finiteness*, preprint, 1976.
2. H. A. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. **81** (1959), 119–159.
3. H. A. Dye, *On groups of measure preserving transformations II*, Amer. J. Math. **85** (1963), 551–576.
4. J. Feldman, P, Hahn and C. Moore, *On groupoids arising from actions of separable locally compact groups*, submitted to Advances in Math.
5. P. H. Forrest, *Virtual subgroups of* $\mathbf{R}^n$ *and* $\mathbf{Z}^n$, Advances in Math. **3** (1974), 187–207.
6. W. Krieger, *On non-singular transformations of a measure space I*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 83–97.
7. W. Krieger, *On the entropy of groups of measure preserving transformations*, unpublished.
8. D. A. Lind, *Locally compact measure preserving flow*, Advances in Math. **15** (1975), 175–193.
9. G. W. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc. **85** (1957), 134–165.
10. G. W. Mackey, *Point realizations of transformation groups*, Illinois J. Math. **6** (1962), 327–335.
11. G. W. Mackey, *Ergodic transformation groups with a pure point spectrum*, Illinois J. Math. **8** (1964), 593–600.
12. C. C. Moore, *Extensions and low dimensional cohomology of locally compact groups I*, Trans. Amer. Math Soc. **113** (1964), 40–63.
13. D. Ornstein, *Bernouilli shifts with the same entropy are isomorphic*, Advances in Math. **4** (1970), 337–352.
14. D. Ornstein and B. Weiss, *The Kakutani–Rohlin Theorem for solvable groups*, preprint.
15. C. Series, *Ergodic actions of product groups*, Harvard Ph. D. Thesis, May 1976.

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